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THE ROLE OF PARABOSE-STATISTICS IN MAKING ABSTRACT QUANTUM THEORY CONCRETE

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Quantum theory is the most fruitful part of physics, no convincing experiment contradicting it has ever been found. The great success of this theory provokes the question:

Can we understand why it is so successful.

The overwhelming success of this theory in experience cannot be deduced from experience. This is so for quantum theory like any other universal law and is known in philosophy already for a long time. Perhaps some people hoped to deduce a theory from sure a priori assumptions. Then such a theory would be unfailing. But - to my knowledge - all those attempts in the past have failed.

Should this indicate not even to ask the question above? I think not. To my knowledge the only convincing attempt for a solution of this problem was made by Kant. This philosopher has clarified that all such general laws **must be necessarily** valid in experience which follow from the **preconditions** of experience alone.

Of course, it is not sure that we are able at all to formulate a priori the preconditions of experience. Then an absolute certaincy for our theories can not be concluded. But it seems to me a useful attempt to look how far it is possible to recognize such preconditions. Then it is a good working hypothesis to use them and to see which other postulates we need to deduce quantum theory from all of them together.

One attempt in this direction of understanding quantum theory and its fundamental role is made by v. Weizsäcker and his co-workers (for an overview see [1-20]) with the concept of "abstract" quantum theory. By abstract quantum theory is designated the general frame of quantum theory, without reference to a three-dimensional position space, to concepts like particle or field, or to special laws of dynamics. Even less it presupposes any set of laws of "classical" physics which would then have to be "quantized".

To understand abstract quantum theory the attempt is made to "reconstruct" it. Reconstruction does not mean a mathematically intended axioma-

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tics but it is the attempt to understand as far as possible the conceptual necessities for quantum theory.

Of course, possesing the frame of abstract quantum theory does not mean that the problems of physics have been solved. The hope of our concept is that beside the principles of abstract quantum theory only one further concept is necessary. This is the idea of introducing the concept of the "ur", the quantized binary alternative.
Then we have to go the long and hard way to understand concrete quantum theory. This means to derive the properties of space-time and of the elementary particles and also of the fundamental forces between them. This is connected with some mathematical problems and also with great conceptual ones. E.g. to derive the properties of space-time from abstract quantum theory means to derive a cosmological model prior to a theory of gravitation. This is done elsewhere ([16], [17], [18]) and will not be referred in this paper. In the present paper a path will be outlined from the abstract concepts of ur theory to the construction of states for real particles.

The reconstruction of abstract quantum theory will be outlined briefly in chapter 1. In chapter 2 we explain the concept of the "ur" and its "second quantization", i.e. the theory of many urs. In the third chapter arguments are given for the use of parabose statistics for urs. The algebra of parabose operators enables us to construct massless and massive particles by the urs. This will be demonstrated on concrete examples in chapter 4.

1. RECONSTRUCTION OF "ABSTRACT QUANTUM THEORY"

"Reconstruction" means the attempt to formulate simple and plausible postulates on prediction and to derive the basic concepts of abstract quantum theory from them. The following procedure is mainly based on the paper "Reconstruction of abstract quantum theory" by Drieschner, v. Weizsäcker and myself (1987), in which the assertions are arranged in four groups: Heuristic principles, verbal definitions, basic postulates, consequences. For the sake of completeness I will give the main points in a short review:

1.1. Heuristic principles

A1. Preconditions of experience: As far as possible our postulates ought to express conditions without which we cannot expect experience to be possible at all.
A2. Simplicity: Without precisely defining simplicity, we wish for simple postulates rather than complicated ones.
A3. Innocuous generality: General rules are usually simpler than specialized ones. We shall confine ourselves to general rules as far as they give the hope of being "innocuous"; e.g. claiming the general existence of a set of states while under special conditions (like a dynamics implying a super-selection-rule) some of those might not actually come into being.

1.2.Definitions

B1. Experience: Experience means to learn from the past for the future.
B2. Facticity of the past: We speak of past events as of objective facts, independently of our actually knowing them.
B3. Possibility of the future: We are aware of future events only as possibilities.

B4. Probability: Probability is a quantification of possibility. We define it as the prediction (mathematically: the expectation value) of a relative frequency.

B5. Temporal statements: A temporal statement (briefly "statement") is a verbal proposition (or a mathematical proposition with a physical meaning) referring to a moment in time.

B6. States: States are recognizable events. A state is what is the case when some temporal statement is true. States at different times can be identical: it is meaningful to ask whether we observe now the same states at a certain time before.

B7. Conditional probability: Let x and y to be two states. Then $p(x,y)$ is the probability that, if x is a present state, y will be found as the state if searched for.

B7. Alternatives: An n-fold alternative is a set of n mutually exclusive states, exactly one of which will turn out to be present if and when an empirical test of this alternative is made.

B8. Connection: Two states x and y are called connected if there is a law of nature determining their conditional probabilities $p(x,y)$ and $p(y,x)$. If the connection is transitive, i.e. if the existence (by law of nature) of probabilities $p(x,y)$ and $p(y,z)$ implies the existence of a $p(x,z)$, then connection is an equivalence relation, defining a partition of the class of all states into subclasses of mutually connected states.

B9. Separability: Two states are called separable if they are not connected.

1.3. Postulates

C1. Separable alternatives:
There are alternatives whose states are separable from nearly all other states. "Nearly" will be defined as meaning all states not connected with the states of the alternative by postulate 2.

C2. Indeterminism:
Let x and y be two connected, mutually exclusive states $\{p(x,y) = p(y,x) = 0\}$, then there are states z, different from x and y, which are not constructed from x and y by mere logical operations and which cannot distinguished by their transformation properties from the former ones but which possess conditional probabilities $p(z,x)$ and $p(z,y)$ none of which is equal to zero or to one. Between every two of the states z there are conditional probabilities $p(z,z')$.

C3. Kinematics:
The conditional probabilities between connected states are not altered when the states change in time:

$$p[(x,t),(z,t)] = p[(x,0),(z,0)].$$

1.4. Consequences

D1. State space: We call the set of states connected with a separable alternative its state space. With innocuous generality we assume the state spaces of all separable n-fold alternatives A_n to be isomorphic: $S(n)$.
A state $z \epsilon S(n)$ defines n conditional probabilities $p(z,x_i)$ where x_i $(i = 1...n)$ are the states defining the n-fold alternative;

$$\sum_{i=1}^{n} p(z,x_i) = 1.$$

D2. Completeness: For any mathematically possible set of values $p(z,x_i)$ there is a state z in S(n). We assume this to be an example of innocuous generality.

D3. Equivalence of states: All states in S(n) are equivalent. Else their distinction would be an additional alternative connected with A_n. A_n would hence not have been separable. (This "separability" should not be confused with the mathematical concept of separability of a Hilbert space)

D4. Symmetry group: The equivalence of the elements of S(n) is expressed by a symmetry group G(n) which preserves the conditional probabilities between them. Due to D3, G(n) must be a continuous group.

D5. Alternatives in S(n): Due to D3 there exists a $p(x,y)$ between any two states x and y of S(n). The equivalence of all states in S(n) further implies that any $z \in S(n)$ is a member of a precisely n-fold alternative of mutually exclusive states of S(n).

D6. Metric in S(n): As an "assumption of simplicity" we suppose G(n) to be a simple Lie group. There are two simple Lie groups preserving a relation of mutual exclusion between precisely n normalized vectors by preserving a metric: O(n) and U(n). Thus we assume S(n) to permit a faithful irreducible representation in an n-dimensional vector space V(n), G(n) being either orthogonal or unitary. The states of S(n) will then correspond to normalized vectors in V(n), i.e. to one-dimensional subspaces.

D7. Dynamics: According to C3, the change of state in time must be a one-parameter subgroup D(t) of G(n). We call the special choice of such a subgroup the choice of a law of dynamics.

D8. Preservation of state: If a state is to be recognizable in time, there must exist a possible law of dynamics which keeps this state constant.

D9. Complexity: The generator of D(t), as defined in D7, must, according to D8, permit diagonalization. This is universally possible only if V is complex, and, due to the metric, a Hilbert space. Hence G(n) = U(n).

D10. Composition: Two alternatives A_m and A_n are simultaneously decided by deciding their Cartesian product $A_{m.n} = A_m \times A_n$. $A_{m.n}$ defines the direct product Hilbert space $V(m.n) = V(m)*V(n)$.

2. THE WAY TO CONCRETE QUANTUM THEORY: THE CONCEPT OF THE "UR"

Traditional quantum theory accepts the concepts of time, space, particle, field, hence of motion, position, momentum, energy, force from classical physics. In the reconstruction only time is used from the outset, and all concepts of objects are replaced by the logical concept of "alternative", all concepts of temporal properties of objects by the concept of "state".

Time, however, is described in a more detailed manner than in classical physics. While it is also measured by a real variable t, explicit use is made of its "modes": present, past, future, with their qualitative differences.

The "abstract" theory (which was outlined above) is general enough to serve as a frame for introducing all the above-mentioned traditional "concrete" terms. All these concepts and the theories referring to them, like relativity and particle theory, should be developed as a consequence of abstract quantum theory and of the addition of one single (and simple) idea: **the reduction of all alternatives to the successive decision of binary alternatives (yes-no decisions, bits or "urs").**

2.1. The "ur" - the quantized binary alternative

It is a trivial fact that every n-fold alternative can be decomposed in a product of binary alternatives and that every state space can be understood as a subspace of a tensor product of two-dimensional spaces.

The central dynamical postulate of the ur hypothesis is:
For any object there is at least one decomposition into binary subobjects - called urs - such that its dynamics is invariant under the symmetry group of the urs.

An ur itself is not something like an "ultimate particle" and not even a "small object". The ur theory is to be understood as a theory of a logical atomism.

The binary (n=2) alternative defines as its state space the space of two complex dimensions $\mathbf{C}^2$, possessing a SU(2) symmetry. Systems defined by a Cartesian product of n binary alternatives possess a state space which is, or is a subspace of, the tensor product of n $\mathbf{C}^2$-spaces:

$$\mathbf{T_n} = \mathbf{C}^2 * \mathbf{C}^2 * \cdots * \mathbf{C}^2$$

All objects with such a type of state space will have at least SU(2) as a symmetry of their dynamics. SU(2) is locally isomorphic to SO(3), and we start from the working hypothesis that this is the reason for a three-dimensional real space offering a natural description of all objects in physics: the "position space". How relativity, and particles as irreducible representations of a relativistic group, can be derived from this hypothesis will be shown in chapter 4.

If the number of urs is not fixed then they must be described in the tensor space $\mathbf{T_B}$ of urs with index set {1,2} and Botzmann statistics:

$$\mathbf{T_B} = \mathbf{C}^2 + \mathbf{C}^2 * \mathbf{C}^2 + \mathbf{C}^2 * \mathbf{C}^2 * \mathbf{C}^2 + \ldots = \mathbf{C}^2 + (\mathbf{C}^2)^{*2} + (\mathbf{C}^2)^{*3} + \ldots + (\mathbf{C}^2)^{*n} + \ldots = \mathop{+}_{1}^{\infty} \mathbf{T_n} \qquad (1)$$

thereby * is the tensor product, + the direct sum and $\mathbf{T_n} = (\mathbf{C}^2)^{*n}$.
In $\mathbf{T_B}$ there is a canonical scalar product induced from $\mathbf{C}^2$. All elements from $\mathbf{T_n}$ are orthogonal to all of $\mathbf{T_m}$ for $n \neq m$.

We start with the assumption that all the urs are different. For the alternatives it must make sense to say which one of it was intended. So as yet no symmetrization is recommended.
If x_1 and x_2 constitute a basis in $\mathbf{C}^2$ then the 2^n monoms

$$x_i \, x_j \, x_k \ldots x_l \, x_m \, x_n \qquad [x_i, x_j, x_k, \ldots, x_l, x_m, x_n \in \{x_1, x_2\}]$$

build up a basis for $\mathbf{T_n}$.
In $\mathbf{T_B}$ it is possible to define mappings between $\mathbf{T_n}$ and $\mathbf{T_{n+1}}$. This can be done by operators for rising or lowering the degree of the tensors. We define l_i^+ , the operator of left-multiplication by x_i , as

$$l_i^+ x = l_i^+ x_j \, x_k \ldots x_l = x_i x_j \, x_k \ldots x_l \qquad (2)$$

l_i^+ is a operator with norm 1 and therefore welldefined on $\mathbf{T_B}$

$$||l_i^+|| = \sup_{||x||=1} ||l_i^+ x|| = 1 \qquad (3)$$

There exists an adjoined operator l_i . It is defined for each y and all $x \in \mathbf{T_B}$ by the condition: There is a y^*

$$y^* = l_i y \quad \text{such that} \ (y, l_i^+ x) = (y^*, x) \ . \qquad (4)$$

l_i removes from x a left hand sided x_i or, if there is non, annihilates x. Therefore it is

$$l_i l_i^+ = \mathrm{id} \tag{5}$$

and

$$l_i^+ l_i = P_i \tag{6}$$

$$l_k l_i^+ = 0 \qquad \text{for } i \neq k . \tag{7}$$

P_i is a projection operator which annihilates all x without a left hand sided x_i.

So l_i^+ and l_i are partial isometries. They can be extended to unitary transformations by an extension of the Hilbert space ([21] pp. 436). This imbedding of $\mathbf{T_B}$ into the space $\mathbf{T}$ can be seen analogously to the imbedding of the natural numbers into the positive and negative whole numbers. In a philosophical sense the whole numbers are to be understood as operations on the natural numbers ([22]). In the same sense the vectors of the extended space $\mathbf{T}$ are operations on the tensor space $\mathbf{T_B}$.

The extension of the partial isometric operators to unitary operators is straight forward. Firstly we have to introduce a onedimensional subspace Ω, the "vacuum", which is defined by

$$l_i x_i = \Omega \quad \text{resp.} \quad l_i^+ \Omega = x_i \quad \text{for every } i \tag{8}$$

and further on more vectors

$$x_{(-i)} = l_i \Omega \quad \text{and so on,} \tag{9}$$

which are orthogonal to all of the vectors from $\mathbf{T_B}$.

In the case of the urs we denote $x_{(-1)}$ by x_4 and $x_{(-2)}$ by x_3,

$$l_1 \Omega \to l_4^+ \Omega \tag{10a}$$

and

$$l_2 \Omega \to l_3^+ \Omega \tag{10b}$$

A basis for the space of all states in $\mathbf{T}$ is given by polynoms in l_i and l_i^+, $i \in \{1,2\}$ (Geyer 1973, p. 180), resp. in l_i^+ alone for $i \in \{1,2,3,4\}$ acting on Ω

$$l_i^+ l_j^+ \ldots l_k^+ l_l^+ \Omega = x_i\, x_j \ldots x_k\, x_l \qquad \text{with } i,j,\ldots,k,l \in \{1,2,3,4\} \tag{11}$$

For this Boltzmann-"ladder"-operators we have

$$l_i^+ l_i = l_i l_i^+ = \mathrm{id} \tag{12a}$$

and therefore

$$[l_i^+, l_i] = 0 \tag{12b}$$

2.2. The introduction of symmetries between the urs

Normally, in the concept of a "second quantization" the ladder operators are used to build up generators for a symmetry group. This group acts on the objects belonging to this new step of quantization.

To construct generators for a symmetry group it is necessary to have commutation relations for the ladder operators. The same sample of vectors constitute different Boltzmann tensors which are differentiated only by the sequences of the vectors in it. Therefore to get commutation relations we have to introduce symmetry relations between such tensors. In the tensors (11) the sequence of the operators is ordered by the place of the operators in the file. To allow symmetry relations between the operators means to weaken the differences of its places in the line.

All states in $\mathbf{T}$ can be expressed by the formula (11). If one wants to diminish the order of the factors, states with both factors x_1 and x_4 or both factors x_2 and x_3 could possibly disappear. Giving away the identityof $l_1\Omega$ and $l_4^+\Omega$ and of $l_2\Omega$ and $l_3^+\Omega$, it becomes possible to preserve the whole manifold of vectors in the extended Boltzmann space as starting point.

We give a new definition

$$l_i \, \Omega = 0 \qquad \text{for every } i \in \{1,2,3,4\} \tag{13}$$

Every x_i in a monom (11) has a fixed position which is not marked in a extra way. To "weaken" the order we label the position by an extra upper index, i.e. we introduce "kinds of urs". Then it becomes possible to give up the demand that all of them must be different. The difference in place is moved into the difference of indices. Therefore adding a new ur, it must be put onto every possible place. The new operators (which are called "Stopf- and Rupf- Operatoren", stuff- and pull-operators [9,11]) are denoted by

$$l_i^+ \dashrightarrow s_i \ , \quad l_i \dashrightarrow r_i \ ,$$

An stuff-operator s_i^a generates an new ur of the "type a" in a state i . It is

$$s_k^b\Omega = x_k^b \tag{14a}$$

and

$$r_i^a \, x_i^a = \Omega \tag{14b}$$

and for any monom in $\mathbf{T}$ (not necessarily only those from tensors generated by some s_i^a out of Ω) it is defined

$$\begin{aligned} s_i^a \, x_j^b x_k^c \ldots x_l^d = N_s[\, & x_i^a x_j^b x_k^c \ldots x_l^d + \\ & +\epsilon(a,b) x_j^b x_i^a x_k^c \ldots x_l^d + \\ & +\epsilon(a,b)\epsilon(a,c) x_j^b x_k^c x_i^a \ldots x_l^d + \ldots \\ \ldots & +\epsilon(a,b)\epsilon(a,c)\ldots\epsilon(a,d) x_j^b x_k^c \ldots x_l^d x_i^a \,] \end{aligned} \tag{15}$$

The normalization factor N_s may depend on the degree of the monom. The sign factor $\epsilon(a,b)$ will be used to differentiate the upper indices:

$$\epsilon(a,b)=1 \text{ for } a=b \ , \ \epsilon(a,b)=-1 \text{ for } a\neq b$$

For the operators r_i^a we get

$$r_i^a \, x_j^b = \delta_{ab}\delta_{ij} \, \Omega \tag{14b}$$

and for any monom in $\mathbf{T}$ (not necessarily generated by some s_i^a out of Ω)

$$\begin{aligned} r_i^a \, x_j^b x_k^c \ldots x_l^d x_m^e = N_r[\, & \delta_{ab}\delta_{ij} \langle x_j^b\rangle x_k^c \ldots x_l^d x_m^e + \\ & +\epsilon(a,b)\delta_{ac}\delta_{ik} x_j^b \langle x_k^c\rangle \ldots x_l^d x_m^e + \ldots \\ \ldots & +\epsilon(a,b)\epsilon(a,c)\ldots\epsilon(\ldots)\delta_{ad}\delta_{il} x_j^b x_k^c \ldots \langle x_l^d\rangle x_m^e + \\ & +\epsilon(a,b)\epsilon(a,c)\ldots\epsilon(a,d)\delta_{ae}\delta_{im} x_j^b x_k^c \ldots x_l^d \langle x_m^e\rangle \,] \end{aligned} \tag{16}$$

whereby $\langle x_j^b\rangle$ means the cancellation of term x_j^b in the monom.

We denote as usual

$$\begin{aligned} s_i^a s_j^b + s_j^b s_i^a &= \{s_i^a, \, s_j^b\} \\ s_i^a s_j^b - s_j^b s_i^a &= [s_i^a, \, s_j^b] \end{aligned} \tag{17}$$

For the operators s_i^a and r_i^a we have the following commutation relations:

$$\{s_i^a, \, s_j^b\} = \{s_i^a, \, r_j^b\} = \{r_i^a, \, r_j^b\} = 0 \quad \text{for } a \neq b \tag{18}$$

and

$$[s_i^a, \, s_j^a] = [r_i^a, \, r_j^a] = 0 \tag{19}$$

2.3. The connection of the ladder operators with the SU(2)-symmetry

A physically meaningful symmetry between the subspaces of $\mathbf{T}$ has to be connected with the symmetry inside the spaces $\mathbf{T}_n$ carrying a tensor representation of SU(2).

For the generators of SU(2) it is

$$[\tau_{jn},\tau_{nl}] = \tau_{jn}\tau_{nl} - \tau_{nl}\tau_{jn} = i\tau_{jl} \qquad (j,n,l = \text{cycl. } \{1,2,3\}) \qquad (20)$$

If u_j und d_k denote in this moment general up- and down- step operators, then for a connection to the SU(2) symmetry we need something like

$$d_k u_j \pm u_j d_k = l^{\pm}_{jk}\, \delta_{jk}\, \mathbf{id} + n^{\pm}_{jk}\tau_{jk} \qquad (21)$$

with numbers $l^{\pm}_{jk}$ and $n^{\pm}_{jk}$ which are to specify.

There are different possibilities for such numbers. E.g. for the demand

$$N_s \equiv 1 \quad \text{and} \quad N_r \equiv 1$$

the operators s_i^a and r_i^a would behave as

$$[r_i^a,\ s_j^a] = \tau_{ij}^a \quad \text{for } i \neq j\ .$$

We denote by n_i^a the number of factors x_i^a in a monom and by

$$n^a = n_1^a + n_2^a + n_3^a + n_4^a \qquad (22)$$

The total number of factors in a monom is

$$n = \Sigma_a\, n^a$$

Then it is

$$[r_i^a,\ s_i^a] = n_i^a + n + 1$$

3. PARABOSE-STATISTICS FOR URS

We will denote by *p* the number of the different types of urs. Till now all the urs are introduced with a clearly distinguishable type. But if the urs can belong with equal probability to every possible type we can get parabose-statistics for the urs. In this case we cannot use any monom in **T**, instead we have to restrict the theory to such tensors which are generated by the s_i out from Ω.

At first we specify the normalization factors N_s and N_r. The commutation and anticommutation relations (18) and (19) are indifferent with respect to N_s and N_r and remain valid. But $[r_i^a,\ s_i^a]$ will be changed.

Now we define operators b_k^{a+} and b_k^a by

$$b_k^{a+} = s_k^a \quad \text{with } N_s = (n^a + 1)^{-1/2} \qquad (23)$$

$$b_k^{a} = r_k^a \quad \text{with } N_r = (n^a\ )^{-1/2} \qquad (24)$$

and get by computation

$$[b_k^a, b_l^{a+}] = \delta_{kl} \qquad [b_k^{a+}, b_l^{a+}] = 0 \qquad [b_k^a, b_l^a] = 0 \qquad (25)$$

and again for $b \neq a$

$$\{b_k^a, b_l^{b+}\} = 0 \qquad \{b_k^{a+}, b_l^{b+}\} = 0 \qquad \{b_k^a, b_l^b\} = 0 \qquad (26)$$

Such anticommuting Bose operators are used in Green's decomposition of parabose operators a_k^+ and a_k ([11] p. 427):

$$a_k^+ = b_{k_1}^{1+} + b_{k_2}^{2+} + \ldots + b_k^{p+} \qquad (27)$$

$$a_k = b_k^1 + b_k^2 + \ldots + b_k^p \qquad (28)$$

It is well known that the a_k^+ and a_k fulfil the three-linear commutation relations

$$[\{a_r, a_s\}, a_t] = 0 \qquad [\{a_r^+, a_s^+\}, a_t^+] = 0 \qquad [\tfrac{1}{2}\{a_r,\ a_s^+\},\ a_t] = -\delta_{st} a_r \qquad (29)$$

The quantum theoretical interpretation of our construction is straight forward. The Green-indizes constitute a hidden classification, any ur belongs with the same probability to any class. The, strictly speaking, required normalization factor $1/p^{1/2}$ is usually neglected in the literature to make the commutation relations free of p.

The connection with the SU(2) symmetry within the subspaces $\mathbf{T_n}$ is given by (see formula (20))

$$\tau_{sr} = (1/2)\{a_r, a_s^+\} \qquad [\tau_{rs}, a_t^+] = \delta_{st} a_r^+ \qquad [\tau_{rs}, a_t] = -\delta_{rt} a_s \tag{30}$$

The combination of two free objects by its tensor product remains valid also for parabose urs. Because of the partial indistinguishability of the urs the tensor product of its state spaces must be constructed as a product of the creation operators over Ω and not of single tensors from $\mathbf{T}$ alone.

By the demand that for different objects the Green's indices should be disjoint we get a new composition rule ("Heidenreich product" [24]). This is different from the tensor product of free objects and should therefore express interaction between the objects ([26], [11] p. 432).

4. PARTICLES WITH PARABOSE-URS

Objects which can move freely in a space-time are normally called particles. Free motion means motion under a maximal group of transformations. In a fourdimensional space-time such a group can be at most 10-dimensional which happens in the case of constant curvature.

By the bilinear combinations of the parabose creation and destruction operators it is possible to construct the representations of SO(4,2) (in connection with the urs see [23, 24, 25]). So it is possible to construct all three groups of motions in a fourdimensional space-time of constant curvature, i.e. SO(4,1), SO(3,2) and the Poincare group. The combination with the linear operators also allows the representations of supersymmetries, connecting Bose- and Fermi- representations.

In ur-theory the cosmological model does not correspond to a space-time with a constant curvature, therefore there does not exist a ten-parameter transformation group. To describe particles we have to go into the tangential space, i.e. Minkowski space-time. There the irreducible representations of the Poincare group give the states of particles.

4.1. The generators of the Poincare group

We use the following abbreviations

$$\tau_{sr} = (1/2)\{a_r, a_s^+\} \tag{31a}$$
$$\alpha^+_{sr} = (1/2)\{a_r^+, a_s^+\} \tag{31b}$$
$$\alpha_{sr} = (1/2)\{a_r, a_s\} \tag{31c}$$

and define the number operator

$$\tilde{n}_r = \tau_{rr} - p/2 \tag{31d}$$

For the generators of O(4,2) (for its Bose form see e.g. [27]) we take the expression given by ([11] p. 407), where a skew-hermitean form of them is used. They are

$$M_{12} = (i/2)(+\tilde{n}_1 - \tilde{n}_2 + \tilde{n}_3 - \tilde{n}_4) \qquad N_{14} = (i/2)(+\alpha_{13} + \alpha^+_{13} - \alpha_{24} - \alpha^+_{24})$$
$$M_{13} = (1/2)(-\tau_{12} + \tau_{21} - \tau_{34} + \tau_{43}) \qquad N_{24} = (1/2)(-\alpha_{13} + \alpha^+_{13} - \alpha_{24} + \alpha^+_{24})$$
$$M_{23} = (i/2)(+\tau_{12} + \tau_{21} + \tau_{34} + \tau_{43}) \qquad N_{34} = (i/2)(-\alpha_{14} - \alpha^+_{14} - \alpha_{23} - \alpha^+_{23})$$

$$M_{15} = (i/2)(+\tau_{12} +\tau_{21} -\tau_{34} -\tau_{43}) \quad N_{16} = (1/2)(-\alpha_{13} +\alpha^+_{13} +\alpha_{24} -\alpha^+_{24})$$
$$M_{25} = (1/2)(+\tau_{12} -\tau_{21} -\tau_{34} +\tau_{43}) \quad N_{26} = (i/2)(-\alpha_{13} -\alpha^+_{13} -\alpha_{24} -\alpha^+_{24}) \quad (32)$$
$$M_{35} = (i/2)(+\tilde{n}_1 -\tilde{n}_2 -\tilde{n}_3 +\tilde{n}_4) \quad N_{36} = (1/2)(+\alpha_{14} -\alpha^+_{14} +\alpha_{23} -\alpha^+_{23})$$

$$M_{46} = (i/2)(+\tilde{n}_1 +\tilde{n}_2 +\tilde{n}_3 +\tilde{n}_4 +2p) \quad N_{46} = (1/2)(+\alpha_{14} -\alpha^+_{14} -\alpha_{23} +\alpha^+_{23})$$
$$s = (1/2)(+\tilde{n}_1 +\tilde{n}_2 -\tilde{n}_3 -\tilde{n}_4) \quad N_{56} = (i/2)(+\alpha_{14} +\alpha^+_{14} -\alpha_{23} -\alpha^+_{23})$$

For the generators of the Poincare group we use the expressions for the translations

$$P_i = M_{i5} + N_{i6} \qquad P_4 = N_{46} + M_{46} \quad (33a)$$

and the operators of SO(3,1) for rotations and Lorentz boosts:

$$M_{12},\ M_{13},\ M_{23}; \qquad N_{14},\ N_{24},\ N_{34} \quad (33b)$$

Then the momentum operators are given explicitly by

$$2P_1 = [i(\tau_{12} +\tau_{21} -\tau_{34} -\tau_{43}) -\alpha_{13} +\alpha_{24} +\alpha^+_{13} -\alpha^+_{24}]$$
$$2iP_2 = [i(\tau_{12} -\tau_{21} +\tau_{43} -\tau_{34}) +\alpha_{13} +\alpha_{24} +\alpha^+_{13} +\alpha^+_{24}] \quad (34)$$
$$2P_3 = [i(\tilde{n}_1 -\tilde{n}_2 -\tilde{n}_3 +\tilde{n}_4) +\alpha_{14} +\alpha_{23} -\alpha^+_{14} -\alpha^+_{23}]$$
$$2P_4 = [i(\tilde{n}_1 +\tilde{n}_2 +\tilde{n}_3 +\tilde{n}_4 +2p) +\alpha_{14} -\alpha_{23} -\alpha^+_{14} +\alpha^+_{23}]$$

4.2. Momentum eigenstates for particles

In ([23, 24, 25]) the representations are given in an abstract way. In the present paper we are interested in concrete momentum eigenstates for particles. This are states $|\Phi\rangle$ which fulfil

$$(P_1+iP_2)|\Phi\rangle = [i(\tau_{12} - \tau_{34}) + \alpha_{24} + \alpha^+_{13}]|\Phi\rangle = 0 \quad (35a)$$
$$(P_1-iP_2)|\Phi\rangle = [i(\tau_{21} - \tau_{43}) - \alpha_{13} - \alpha^+_{24}]|\Phi\rangle = 0 \quad (35b)$$
$$(P_4+P_3)|\Phi\rangle = [i(\tilde{n}_1 + \tilde{n}_4 + p) + \alpha_{14} - \alpha^+_{14}]|\Phi\rangle = im_+|\Phi\rangle \quad (35c)$$
$$(P_4-P_3)|\Phi\rangle = [i(\tilde{n}_2 + \tilde{n}_3 + p) - \alpha_{23} + \alpha^+_{23}]|\Phi\rangle = im_-|\Phi\rangle \quad (35d)$$

If m_+ and m_- are both different from zero we have a state of a massive particle and for $m_+m_-=0$ a massless one. A state with $m_+=m_-=0$ is a vacuum state.

For a state of a massless and spinless boson we make the ansatz

$$|\Phi\rangle = \Sigma_\mu \Sigma_\beta\ (-1)^{\mu+\beta} i^{\mu-\beta} (\mu!\beta!)^{-1} c(\mu,\beta) {\alpha^+_{14}}^{\mu} {\alpha^+_{23}}^{\beta}\ |\Omega\rangle \quad (36)$$

and get under the condition $m_+ \neq 0$ for the coefficients $c(\mu,\beta)$ the equations

$$c(\mu,\beta)=c(\mu) \quad (37a)$$

and

$$(\mu+p)c(\mu+1) - (2\mu+p-m_+)c(\mu) - \mu c(\mu-1) = 0 \quad (37b)$$

For $p=1$, formula (37b) is the recurrence relation for the Laguerre polynomials.

A state of a massless boson with helicity σ is constructed from

$$|\phi\rangle = \Sigma_\mu \Sigma_\beta\ i^{\beta-\mu} (\beta!\mu!)^{-1} c(\mu,\sigma)\ {\alpha^+_{11}}^{\sigma} {\alpha^+_{14}}^{\mu} {\alpha^+_{23}}^{\beta}\ |\Omega\rangle \quad (38)$$

For the condition $m_+ \neq 0$ the coefficients $c(\mu,\sigma)$ must fulfil the equations

$$(\mu+p+2\sigma)c(\mu+1,\sigma) - (2\mu+p+2\sigma-m_+)c(\mu,\sigma) + \mu c(\mu-1,\sigma) = 0 \quad (39)$$

A state of a massless fermion with helicity 1/2 is realized by

$$|\Gamma\rangle = \Sigma_\mu \Sigma_\beta\ i^{\beta-\mu} (\beta!\mu!)^{-1} c(\mu)\ a^+_1 {\alpha^+_{14}}^{\mu} {\alpha^+_{23}}^{\beta}\ |\Omega\rangle \quad (40)$$

with $m_+ \neq 0$ and

$$+(\mu+p+1)c(\mu+1) - (2\mu+p+1-m_+)c(\mu) + \mu c(\mu-1) = 0 \quad (41)$$

It is obvious that for such massless particles these states are nearly of the same form.

Now we have to look for massive particles.
Massive particles cannot made from power series in $\alpha^{+}{}_{14}{}^{\mu}$ and $\alpha^{+}{}_{23}{}^{ß}$ alone, there must be extra factors of the type $\alpha^{+}{}_{13}{}^{\sigma}$ and $\alpha^{+}{}_{24}{}^{\sigma}$.

For a massive spinless particle we made the the ansatz

$$|\Theta\rangle = \Sigma_{\sigma}\Sigma_{\mu}\Sigma_{ß} i^{ß-\mu}(\mu!ß!\sigma!\sigma!)^{-1} g(\mu,ß,\sigma)\alpha^{+}{}_{13}{}^{\sigma}\alpha^{+}{}_{24}{}^{\sigma}\alpha^{+}{}_{14}{}^{\mu}\alpha^{+}{}_{23}{}^{ß}\ |\Omega\rangle \qquad (42)$$

and get from (35a-d) the conditions

$$(\sigma+1)g(\mu,ß,\sigma) +(\sigma+1)g(\mu+1,ß+1,\sigma) -(\sigma+1)g(\mu,ß+1,\sigma) -(\sigma+1)g(\mu+1,ß,\sigma)$$
$$-\mu g(\mu-1,ß,\sigma+1) -ßg(\mu,ß-1,\sigma+1) +(\mu+ß+p+\sigma)g(\mu,ß,\sigma+1)= 0 \qquad (43)$$

$$-[2\mu+2\sigma+p-m_{+}]g(\mu,ß,\sigma) +(2\sigma+\mu+p)g(\mu+1,ß,\sigma) +\mu g(\mu-1,ß,\sigma)$$
$$= -ßg(\mu,ß-1,\sigma+1) \qquad (44)$$

$$-[2ß+2\sigma+p-m_{-}]g(\mu,ß,\sigma) +(2\sigma+p+ß)g(\mu,ß+1,\sigma) +ßg(\mu,ß-1,\sigma)$$
$$= -\mu g(\mu-1,ß,\sigma+1) \qquad (45)$$

from this equations it follows

$$-[2\mu+2\sigma+p-m_{+}]g(\mu,ß+1,\sigma) +(2\sigma+\mu+p)g(\mu+1,ß+1,\sigma) +\mu g(\mu-1,ß+1,\sigma)$$
$$= -(ß+1)g(\mu,ß,\sigma+1) \qquad (46)$$

$$-[2ß+2\sigma+p-m_{-}]g(\mu+1,ß,\sigma) +(2\sigma+p+ß)g(\mu+1,ß+1,\sigma) +ßg(\mu+1,ß-1,\sigma)$$
$$= -(\mu+1)g(\mu,ß,\sigma+1) \qquad (47)$$

and we can get an equation with fixed σ:

$$(\mu+1)\{-[2\mu+2\sigma+p-m_{+}]g(\mu,ß+1,\sigma)+(2\sigma+\mu+p)g(\mu+1,ß+1,\sigma)+\mu g(\mu-1,ß+1,\sigma)\} =$$
$$= (ß+1)\{-[2ß+2\sigma+p-m_{-}]g(\mu+1,ß,\sigma)+(2\sigma+ß+p)g(\mu+1,ß+1,\sigma)+ßg(\mu+1,ß-1,\sigma)\} \qquad (48)$$

If we want to solve it by separation of variables we get in any case mass zero equations. It seems only possible to solve it step by step. In this case we get with the ansatz

$$g(0,0,0) = 1$$

the following values for the coefficients $g(\mu,ß,\sigma)$

$$g(1,0,0) = (p-m_{+})/p$$
$$g(0,1,0) = (p-m_{-})/p$$
$$g(0,0,1) = m_{+}m_{-}/p(p+1)(p-1)$$
$$g(2,0,0) = [(p-2m_{+})/p +m_{+}^{2}/p(p+1)]$$
$$g(1,1,0) = [(p-m_{+}-m_{-})/p +m_{+}m_{-}/(p+1)(p-1)]$$
$$g(0,2,0) = [(p-2m_{-})/p +m_{-}^{2}/p(p+1)]$$
$$g(0,1,1) = m_{+}m_{-}(2+p-m_{-})/p(p+1)(2+p)(p-1)$$
$$= m_{+}m_{-}/p(p+1)(p-1) -m_{+}m_{-}^{2}/p(p+1)(2+p)(p-1)$$
$$g(1,0,1) = m_{+}m_{-}(2+p-m_{+})/p(p+1)(2+p)(p-1)$$
$$= m_{+}m_{-}/p(p+1)(p-1) -m_{-}m_{+}^{2}/p(p+1)(2+p)(p-1)$$
$$g(0,0,2) = m_{+}m_{-}/p(p^{2}-1)$$
$$g(1,1,1) = [(3+p-m_{+}-m_{-})p +(2-2m_{+}-2m_{-}+m_{-}m_{+})m_{+}m_{-}]/p(2+p)^{2}(p^{2}-1)$$
$$g(2,1,0) = [(p-2m_{+})/p +m_{+}^{2}/p(1+p) -m_{-}(p+1-m_{+})/p(1+p)$$
$$+m_{+}m_{-}(2+p-m_{+})(p^{2}+2p-1)/p(1+p)(2+p)(p^{2}-1)]$$
$$g(1,2,0) = [(p-2m_{-})/p +m_{-}^{2}/p(1+p) -m_{+}(p+1-m_{-})/p(1+p)$$
$$+m_{+}m_{-}(2+p-m_{-})(p^{2}+2p-1)/p(1+p)(2+p)(p^{2}-1)]$$
$$g(0,2,1) = (6+5p-6m_{-}+p^{2}-2pm_{-}+m_{-}m_{-})m_{+}m_{-}/p(2+p)(3+p)(p^{2}-1)$$

and so on.

4.3. The introduction of the Lorentz vacuum

The last formulas became very difficult, no closed solution seems to be possible. Therefore it is reasonable to look for a simplification. This can be got by introducing the Lorentz vacuum $|\Omega_L>$.

The states given above are constructed over the ur-vacuum. For this there is no ur being present, i. e. it posses not any bit of information. The meaning of the Lorentz vacuum is that there is not any real particle. The knowledge "there is no particle" is of much greater amount of information than the knowledge "there is no ur". Therefore it seems plausible and also easier if we construct the particle states over the Lorentz vacuum instead the ur vacuum.

We define the Lorentz vacuum by the equations

$$(P_1+iP_2)|\Omega_L> = [i(\tau_{12} - \tau_{34}) + \alpha_{24} + \alpha^+_{13}]|\Omega_L> = 0 \quad (49a)$$
$$(P_1-iP_2)|\Omega_L> = [i(\tau_{21} - \tau_{43}) - \alpha_{13} - \alpha^+_{24}]|\Omega_L> = 0 \quad (49b)$$
$$(P_4+P_3)|\Omega_L> = [i(\tilde{n}_1 + \tilde{n}_4 + p) + \alpha_{14} - \alpha^+_{14}]|\Omega_L> = 0 \quad (49c)$$
$$(P_4-P_3)|\Omega_L> = [i(\tilde{n}_2 + \tilde{n}_3 + p) - \alpha_{23} + \alpha^+_{23}]|\Omega_L> = 0 \quad (49d)$$

which have the solution

$$|\Omega_L> = \Sigma_\mu \Sigma_\beta \, (-1)^{\mu+\beta} i^{\mu-\beta} (\mu!\beta!)^{-1} \alpha^+_{14}{}^{\mu} \alpha^+_{23}{}^{\beta} \, |\Omega> = exp\ i(\alpha^+_{23} - \alpha^+_{14})|\Omega> \quad (50)$$

Now we are able to create particles out from the Lorentz vacuum instead out from the ur vacuum. The formulas for the particle states became much simpler.

4.4. Massless particles from the Lorentz vacuum

We start with a particle of mass zero and spin σ. Its state

$$\Phi(m_+,\sigma)^+|\Omega_L>$$

is defined by the conditions

$$(P_1+iP_2)\Phi(m_+,\sigma)^+|\Omega_L> = 0 \quad (51a)$$
$$(P_1-iP_2)\Phi(m_+,\sigma)^+|\Omega_L> = 0 \quad (51b)$$
$$(P_4-P_3)\ \Phi(m_+,\sigma)^+|\Omega_L> = 0 \quad (51c)$$
$$(P_4+P_3)\Phi(m_+,\sigma)^+|\Omega_L> = im_+\Phi(m_+,\sigma)^+|\Omega_L> \quad (51d)$$
$$M_{12}\ \Phi(m_+,\sigma)^+|\Omega_L> = i\sigma\Phi(m_+,\sigma)^+|\Omega_L> \quad (51e)$$

The ansatz

$$\Phi(m_+,\sigma)^+|\Omega_L> = \alpha^+_{11}{}^{\sigma} \sum_{\mu=0}^{\infty} c(\mu,\sigma)\alpha^+_{14}{}^{\mu}|\Omega_L> \quad (52)$$

results in

$$c(\mu,\sigma)= c(0,\sigma)(im_+)^\mu \frac{(p+2\sigma-1)!}{\mu!(p+2\sigma-1+\mu)!} \quad (\text{for } \mu=0,1,2,...) \quad (53)$$

For spin-1/2-particles we have to replace $\alpha^+_{11}{}^{\sigma}$ by $a^+_1\alpha^+_{11}{}^{\sigma}$ in formula (52) and in (53) σ by $(\sigma+1/2)$.

With the formulas (52) and (53) we have given the momentum eigenstates for massless Bosons and Fermions as expansion series in the parabose operators.

4.5. Massive spinless particles from the Lorentz vacuum

Massive Bosons without spin must fulfil the conditions

$$(P_1+iP_2)\Phi(m)^+|\Omega_L> = 0 \quad (54a)$$
$$(P_1-iP_2)\Phi(m)^+|\Omega_L> = 0 \quad (54b)$$
$$(P_4+P_3)\Phi(m)^+|\Omega_L> = im_+\Phi(m)^+|\Omega_L> \quad (54c)$$
$$(P_4-P_3)\Phi(m)^+|\Omega_L> = im_-\Phi(m)^+|\Omega_L> \quad (54d)$$

with $m_-m_+\neq 0$. They can be constructed from the ansatz

$$\Phi(m)^+|\Omega_L> = \Sigma_\sigma \Sigma_\mu \Sigma_\beta \, h(\mu,\beta,\sigma)\alpha^+_{13}{}^\sigma \alpha^+_{24}{}^\sigma \alpha^+_{14}{}^\mu \alpha^+_{23}{}^\beta \, |\Omega_L> \quad (55)$$

for which (54) give the conditions

$$-(\mu+1)(\beta+1)h(\mu+1,\beta+1,\sigma) \; -(\sigma+1)(p+\sigma+2)h(\mu,\beta,\sigma+1)=0$$
$$-im_+h(\mu,\beta,\sigma) \; +(\sigma+1)^2h(\mu,\beta-1,\sigma+1) \; +(\mu+1)(p+\mu+2\sigma)h(\mu+1,\beta,\sigma)=0$$
$$-im_-h(\mu,\beta,\sigma) \; -(\sigma+1)^2h(\mu-1,\beta,\sigma+1) \; -(\beta+1)(p+\beta+2\sigma)h(\mu,\beta+1,\sigma)=0$$

The coefficients of the solution have the following recurrent relations

$$h(\mu+1,\beta,\sigma)= h(\mu,\beta,\sigma)\frac{im_+}{(\mu+1)} * \frac{(p+\sigma+2)}{[(p+\mu+2\sigma)(p+\sigma+2)-\beta(\sigma+1)]} \quad (\beta>0) \quad (56)$$

$$h(\mu,\beta+1,\sigma)= h(\mu,\beta,\sigma)\frac{-im_-}{(\beta+1)} * \frac{(p+\sigma+2)}{[(p+\beta+2\sigma)(p+\sigma+2)-\mu(\sigma+1)]} \quad (\mu>0) \quad (57)$$

$h(\mu,\beta,\sigma+1)=$ {for $\mu>0$, $\beta>0$, $\mu\neq\beta$}

$$= \frac{h(\mu,\beta,\sigma) \; (-m_+m_-) \; (p+\sigma+2) \qquad (p+\beta+\mu+2\sigma)}{(\sigma+1)[(p+\sigma+2)(p+\beta+2\sigma)-\mu(\sigma+1)][(p+\sigma+2)(p+\mu+2\sigma)-\beta(\sigma+1)](p+\mu+\beta+2\sigma+1)} \quad (58)$$

and {for $\mu>0$, $\beta>0$, $\mu=\beta$}

$$h(\mu,\mu,\sigma+1)= \frac{h(\mu,\mu,\sigma) \; (-m_+m_-) \; (p+\sigma+2)}{(\sigma+1)[(p+\mu+2\sigma)(p+\sigma+2)-\mu(\sigma+1)][(p+\mu+2\sigma)(p+\sigma+2)-(\mu+1)(\sigma+1)]} \quad (59)$$

Massive particles with spin are to describe by more complicated formulas. But in principle they are not different from the former ones.

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